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A BGK Model for Small Prandtl Number in the Navier–Stokes Approximation

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We present a BGK-type collision model which approximates, by a Chapman–Enskog expansion, the compressible Navier–Stokes equations with a Prandtl number that can be chosen arbitrarily between 0 and 1. This model has the basic properties of the Boltzmann equation, including the H-theorem, but contains an extra parameter in comparison with the standard BGK model. This parameter is introduced multiplying the collision operator by a nonlinear functional of the distribution function. It is adjusted to the Prandtl number.

KEY WORDS: Boltzmann equation; compressible Navier–Stokes equations; Prandtl number; entropy.

1. INTRODUCTION

The most accepted model which describes the evolution of the density of a rarefied gas is the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \tag{1}$$

where $f(t, x, v)$ is the distribution function, t the time variable, x the space variable, and v the velocity variable. Because of the complexity of the Boltzmann collision term $Q(f, f)$, many authors tried to substitute a simpler operator, as in the BGK model introduced by Bhatnagar, Gross, and Krook[(2)] or Welander,[(17)] where the operator Q is replaced by

$$J(f) = \nu(M[f] - f) \tag{2}$$

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Usually ν is taken proportional to the macroscopic density $\rho(t, x)$ and $M[f]$ is the Maxwellian associated to f (see below for precise definitions). This model equation preserves the basic properties of the Boltzmann equation: conservation of mass, momentum, and energy, and dissipation of entropy. It also gives, in the fluid limit, the Euler equations.

In this paper we are interested in the Navier–Stokes equations derived from (2) through a Chapman–Enskog expansion.[5] Many authors have worked on this subject for the Boltzmann equation (see, for instance, refs. 1, 3, 5, 8, 9, and 16) and it is well known that the resulting Prandtl number (for a monatomic gas) is $\mathrm{Pr}=2/3$. For the BGK model, Cercignani[4] pointed out that the associated Prandtl number is 1 and consequently raised the problem of finding a model which would give other Prandtl numbers while keeping the H-theorem.

There are other motivations for this question. First, it is not easy to find in the Boltzmann equation (1) the cross section adapted to a given Prandtl number. Second, such a model could have numerical applications, as in the work of Pullin[13] and Reitz,[14] since the BGK model is simpler to approximate than the Boltzmann equation.

We now present such a model. We introduce a multiplier $\lambda_f(t, x, v)$, while $\nu = \nu(t, x)$, and we set

$$\partial_t f + v \cdot \nabla_x f + \nu \lambda_f (f - M) = 0 \tag{3}$$

where

$$M := M[f] = \frac{\rho}{(2\pi T)^{N/2}} e^{-|v-u|^2/2T} \tag{4}$$

is the Maxwellian of density $\rho(t, x)$, mean velocity $u(t, x)$, and temperature $T(t, x)$. These macroscopic quantities are chosen as follows. First, we define ρ_f, u_f, and T_f by

$$(\rho_f, \rho_f u_f, \rho_f(|u_f|^2 + NT_f)) = \int_{\mathbb{R}^N} (1, v, |v|^2) f(t, x, v)\, dv \tag{5}$$

and

$$\lambda_f(t, x, v) = \lambda_0 \left(\frac{v - u_f(t, x)}{[T_f(t, x)]^{1/2}} \right) \tag{6}$$

Then ρ, u, and T are chosen so that

$$\int_{\mathbb{R}^N} \lambda_f M[f](1, v, |v|^2)\, dv = \int_{\mathbb{R}^N} \lambda_f f(1, v, |v|^2)\, dv \tag{7}$$

In fact it will turn out that (ρ_f, u_f, T_f) and (ρ, u, T) are ε^2 close and thus, for the Navier–Stokes approximation, these two macroscopic variables are equivalent.

Finally, we will choose

$$v(t, x) = \rho_f \phi(T_f) \tag{8}$$

and ϕ is related to the behavior in T of the viscosity and heat conduction terms of Navier–Stokes equations.

The Prandtl number Pr is governed by the choice of $\lambda_0(\cdot)$. Here we assume

$$\exists C_1, C_2, \quad 0 < C_1 \leqslant \lambda_0(w) \leqslant C_2 < +\infty \tag{9}$$

$$\lambda_0(w) = \lambda_0(|w|) \tag{10}$$

$$\int_{\mathbb{R}^N} \frac{M_0(w)}{\lambda_0(w)}\, dw = 1, \qquad \int \frac{M_0}{\lambda_0} |w|^2\, dw = N,$$

$$\int \frac{M_0}{\lambda_0} |w|^4\, dw = N(N+2) \tag{11}$$

Then

$$\int \frac{M_0}{\lambda_0} |w|^6\, dw = \frac{2N(N+2)}{\Pr} + N(N+2)^2 \tag{12}$$

We have used the notation

$$M_0(w) = \frac{e^{-|w|^2/2}}{(2\pi)^{N/2}} \tag{13}$$

A difference from the model proposed in ref. 4 is that, as the mean free path ε goes to 0, f converges to a Maxwellian, thus recovering the classical density limit.

The rest of the paper is organized as follows. In Section 2 we derive the relation (12) for the Prandtl number in the Navier–Stokes approximation. Then we prove in Section 3 that we can indeed compute (ρ, u, T) in a unique way from the relation (7). We also introduce a "variational" principle which gives the H-theorem. Section 4 is devoted to an existence and stability proof for Eq. (3) when $v = 1$, using the averaging lemma. Finally, we show in Section 5 that the whole interval $]0, 1]$ for the Prandtl number can be reached by appropriate choices of λ_0.

2. DERIVATION OF THE NAVIER–STOKES EQUATIONS

In this section, we show the relationship between the kinetic model and the hydrodynamic equations. We adopt a derivation which differs slightly from the classical Chapman–Enskog expansion.[4,5,16] We have found this presentation in Deshpande[6] and it seems simpler for our purpose.

This section is organized as follows. We first recall how to recover the Navier–Stokes equations. Then we prove that $(\rho, u, T) = (\rho_f, u_f, T_f) + O(\varepsilon^2)$. This allows us to compute the Navier–Stokes coefficients deduced from our BGK model. Then we conclude this section with some remarks.

2.1. The Navier–Stokes Equations

We consider a solution f of the equation

$$\partial_t f + v \cdot \nabla_x f = \rho_f \lambda_f \frac{M[f] - f}{\varepsilon} := g \tag{14}$$

with the definitions (4)–(7). Since g satisfies

$$\int_{\mathbb{R}^N} (1, v, |v|^2)\, g\, dv = 0$$

the macroscopic quantities ρ_f, u_f, T_f satisfy the system

$$\partial_t \rho_f + \operatorname{div}(\rho_f u_f) = 0 \tag{15}$$

$$\partial_t(\rho_f u_f) + \operatorname{div}(\rho_f u_f \otimes u_f + P) = 0 \tag{16}$$

$$\partial_t \left[\rho_f \left(\frac{|u_f|^2}{2} + \frac{N T_f}{2} \right) \right] + \operatorname{div} \left[\rho_f \left(\frac{|u_f|^2}{2} + \frac{N T_f}{2} \right) u_f + P u_f + Q \right] = 0 \tag{17}$$

The matrix P and the vector Q are defined by

$$P_{ij} = \int (v - u_f)_i (v - u_f)_j f\, dv \tag{18}$$

$$q_i = \tfrac{1}{2} \int |v - u_f|^2 (v - u_f)_i f\, dv \tag{19}$$

Finally, $u \otimes u$ denotes the symmetric matrix with coefficients $u_i u_j$ and $\operatorname{div}(u \otimes u)$ the vector of components $\sum_j \partial(u_i u_j)/\partial x_j$.

The system (15)–(17) is the compressible Navier–Stokes system for a monatomic gas if

$$P = \rho_f T_f I_N - \mu(T_f)\left(\sigma - \frac{2}{N} I_N \operatorname{div} u_f\right) \tag{20}$$

$$Q = -\kappa(T_f)\nabla T_f \tag{21}$$

where I_N is the identity matrix and

$$\sigma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{22}$$

The Prandtl number is the ratio

$$\Pr = \frac{N+2}{2}\frac{\mu(T)}{\kappa(T)} \tag{23}$$

2.2. A Preliminary Result

In order to compute P and Q up to ε^2 (as ε goes to 0) when f satisfies the BGK equation (14), we just notice that

$$f - M = O(\varepsilon) \tag{24}$$

and thus, using (14) again,

$$\begin{aligned} f - M &= -\frac{\varepsilon}{\rho_f \lambda_f}(\partial_t f + v \cdot \nabla_x f) \\ &= -\frac{\varepsilon}{\rho_f \lambda_f}(\partial_t M + v \cdot \nabla_x M) + O(\varepsilon^2) \end{aligned} \tag{25}$$

We will show that replacing f in (18) and (19) by the expression given in terms of M through (25) yields the appropriate form of P and Q.

We need a preliminary result:

Lemma 1. We have

$$\int_{\mathbb{R}^N} (1, v, |v|^2)(f - M)\, dv = O(\varepsilon^2) \tag{26}$$

In other words, (ρ_f, u_f, T_f) differs from (ρ, u, T) by a term of order $O(\varepsilon^2)$.

The proof of this lemma (which is of course formal) consists in first noticing that, thanks to (25), we have

$$f - M = O(\varepsilon)$$

therefore

$$\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} (\partial_t M + v \cdot \nabla_x M)\, dv$$

$$= O(\varepsilon) + \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} (\partial_t f + v \cdot \nabla_x f)\, dv = O(\varepsilon) \tag{27}$$

But

$$\partial_t M + v \cdot \nabla_x M = \frac{M}{\rho} (\partial_t \rho + v \cdot \nabla_x \rho) + \frac{v-u}{T} M(\partial_t u + v \cdot \nabla_x u)$$
$$+ \frac{M}{2T} \left(\frac{|v-u|^2}{T} - N \right) (\partial_t T + v \cdot \nabla_x T) \tag{28}$$

and thus the lhs of (27) only depends on the five first moments of M. By the assumptions (10), (11) on λ_0, this gives also

$$\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \frac{1}{\lambda_f} (\partial_t M + v \cdot \nabla_x M)\, dv$$

$$= \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \frac{1}{\lambda_0((v-u)/\sqrt{T})} (\partial_t M + v \cdot \nabla_x M)\, dv + O(\varepsilon) = O(\varepsilon) \tag{29}$$

Combining this estimate with (25) gives the result announced in Lemma 1.

2.3. Computation of *P*

The matrix P is easy to compute; it only requires the five first velocity moments of M. Setting $\lambda := \lambda_0((v-u)/\sqrt{T})$, we have

$$
\begin{aligned}
P_{ij} &= \int_{\mathbb{R}^N} (v_i - u_f^i)(v_j - u_f^j)\, f\, dv \\
&= \int_{\mathbb{R}^N} (v_i - u_i)(v_j - u_j)\, f\, dv + O(\varepsilon^2) \\
&= \rho T \delta_{ij} + \int_{\mathbb{R}^N} (v_i - u_i)(v_j - u_j)(f - M)\, dv + O(\varepsilon^2) \\
&= \rho T \delta_{ij} - \frac{\varepsilon}{\rho} \int_{\mathbb{R}^N} (v_i - u_i)(v_j - u_j) \\
&\quad \times \frac{1}{\lambda} (\partial_t M + v \cdot \nabla_x M)\, dv + O(\varepsilon^2)
\end{aligned}
$$

The integral term in the above formula can be evaluated in the usual way and we obtain

$$
P_{ij} = \rho T \delta_{ij} - \varepsilon T \left(\sigma_{ij} - \frac{2}{N} \delta_{ij} \operatorname{div} u \right) + O(\varepsilon^2) \tag{30}
$$

2.4. Computation of Q

Up to an $O(\varepsilon^2)$ term, we have

$$
\begin{aligned}
2\rho q_i &= -\varepsilon \int_{\mathbb{R}^N} |v - u|^2 (v_i - u_i) \frac{1}{\lambda} (\partial_t M + v \cdot \nabla_x M)\, dv \\
&= -\varepsilon \int_{\mathbb{R}^N} (v_i - u_i)\, |v - u|^2 \frac{M}{\rho \lambda} (v - u) \cdot \nabla_x \rho\, dv \\
&\quad - \varepsilon (\partial_t u + u \cdot \nabla_x u) \int_{\mathbb{R}^N} (v_i - u_i)\, |v - u|^2 \frac{v - u}{T\lambda} M\, dv \\
&\quad - \varepsilon \int_{\mathbb{R}^N} (v_i - u_i)\, |v - u|^2 \frac{M}{2\lambda T} \left(\frac{|v - u|^2}{T} - N \right) \\
&\quad \times (v - u) \cdot \nabla_x T\, dv \\
&= -\varepsilon \partial_{x_i} \rho \int_{\mathbb{R}^N} (v_i - u_i)^2\, |v - u|^2 \frac{M}{\rho \lambda}\, dv \\
&\quad - \varepsilon \frac{1}{T} (\partial_t u_i + u \cdot \nabla_x u_i) \int_{\mathbb{R}^N} (v_i - u_i)^2\, |v - u|^2 \frac{M}{\lambda}\, dv \\
&\quad - \varepsilon \frac{1}{2} \partial_{x_i} T \int (v_i - u_i)^2 \frac{|v - u|^2}{T} \left(\frac{|v - u|^2}{T} - N \right) \frac{M}{\lambda}\, dv \\
&= -\varepsilon (N + 2)\, T^2 \partial_{x_i} \rho - \varepsilon (N + 2)\, \rho T (\partial_t u_i + u \cdot \nabla_x u_i) \\
&\quad - \varepsilon \frac{1}{2} \partial_{x_i} T \left[\frac{\rho T}{N} I_6 - \rho T N (N + 2) \right]
\end{aligned}
$$

where

$$I_6 = \int_{\mathbb{R}^N} |w|^6 \frac{M_0(w)}{\lambda_0(w)} \, dw$$

But the momentum equation in the Euler system gives

$$T \, \partial_{x_i} \rho + \rho(\partial_t u_i + u \cdot \nabla_x u_i) = -\rho \, \partial_{x_i} T + O(\varepsilon)$$

and we finally obtain

$$\begin{aligned} q_i &= -\frac{\varepsilon}{2}\left(\frac{I_6}{2N} - \frac{(N+2)^2}{2}\right) T \, \partial_{x_i} T + O(\varepsilon^2) \\ &= -\frac{N+2}{2 \, \mathrm{Pr}} \varepsilon T \frac{\partial T}{\partial x_i} + O(\varepsilon^2) \end{aligned} \tag{31}$$

where Pr is given by (12). Hence we recover the Navier–Stokes equations up to an $O(\varepsilon^2)$ term, as is usual in the Chapman–Enskog expansion. Moreover, the viscosity and heat conduction terms have the coefficients

$$\mu(T) = \varepsilon T, \qquad \kappa(T) = \varepsilon \frac{N+2}{2 \, \mathrm{Pr}} T \tag{32}$$

which means that the formulas (23) and (12) for the Prandtl number are equivalent.

2.5. Temperature Law

It is easy to modify the dependence of μ and κ on T [see (32)]. To do so, we just have to modify the BGK model (14) setting

$$\partial_t f + v \cdot \nabla_x f + \rho_f \chi(T_f)^{-1} \lambda_f \frac{f - M}{\varepsilon} = 0$$

The same calculations as before go through and give

$$\mu(T) = \varepsilon T \chi(T), \qquad \kappa(T) = \varepsilon \frac{N+2}{2 \, \mathrm{Pr}} T \chi(T)$$

These provide general laws for the coefficients $\mu(T)$ and $\kappa(T)$, which are necessary to treat high-temperature flows.

Remark. The Navier–Stokes equation can be formally written, using the above calculations,

$$\int \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} (\partial_t + v \cdot \nabla_x) \left[f - \frac{\varepsilon}{\rho\lambda} (\partial_t f + v \cdot \nabla_x f) \right] dv = O(\varepsilon^2)$$

with f a Maxwellian. This equation involves a diffusion term which depends upon ρ, u, T. A Boltzmann model has been proposed by Klimontovich[10] on a physical basis which also contains such diffusion terms.

3. EXISTENCE OF A MAXWELLIAN WITH THE RIGHT VELOCITY MOMENTS. VARIATIONAL PRINCIPLE

In this section we show that it is always possible to find, once the parameters u_f and T_f of λ are fixed, a unique Maxwellian which satisfies the relations

$$\int_{\mathbb{R}^N} (1, v, |v|^2)\, \lambda M\, dv = (\alpha, \alpha\beta, \alpha(|\beta|^2 + N\gamma)) \tag{33}$$

for given $\alpha \geqslant 0$, $\beta \in \mathbb{R}^N$, $\gamma > 0$. This is necessary for our BGK model (14), which requires we solve (7). We also show that, associated to (33), M satisfies a variational principle of Gibbs type.

Concerning the resolution of (33), it is enough to consider $\lambda(v)$ as any function of $|v - u_0|$, for some $u_0 \in \mathbb{R}^N$, which satisfies (9).

Proposition 2. Let $\lambda := \lambda(|v - u_0|)$ satisfy (9); then, for any parameters $\alpha \geqslant 0$, β and $\gamma > 0$ there exists a unique Maxwellian M satisfying (33).

Proof. We only treat the case $\alpha > 0$, since $\alpha = 0$ is uniquely solved by $M = 0$. We first have to fix ρ by

$$\alpha/\rho = \int \chi\, dv$$

$$\chi(v) = \frac{\lambda(v)}{(2\pi T)^{N/2}} e^{-|v-u|^2/2T} \tag{34}$$

and the problem is reduced to finding u and T such that

$$\int v\chi\, dv = \beta \int \chi\, dv, \qquad \int |v|^2 \chi\, dv = (|\beta|^2 + N\gamma) \int \chi\, dv \tag{35}$$

The first equation can be reduced to a scalar problem and solved as a function $u(\beta, T)$.

To prove it, we define

$$V(T, u) = \int v\chi \, dv \Big/ \int \chi \, dv$$

and we will show that, for a fixed T, V is a C^1 diffeomorphism of $\mathbb{R}^N$, whose inverse $W(T, \cdot)$ is C^1 of its two arguments.

To do so, we set

$$u = u_0 + rw, \qquad |w| = 1, \qquad r \geqslant 0$$

and the definition of χ in (34) gives

$$\begin{aligned} V &= u_0 + w \int \chi(v - u_0) \cdot w \, dv \Big/ \int \chi \, dv \\ &= u_0 + wR(r) \end{aligned}$$

$R(r)$ does not depend on w and satisfies

$$\begin{aligned} \frac{dR}{dr} &= \Big[\int \chi(v - u_0) \cdot w(v - u) \cdot w \int \chi \\ &\quad - \int \chi(v - u_0) \cdot w \int \chi(v - u) \cdot w \Big] \Big/ \Big(T^{1/2} \int \chi \Big)^2 \\ &= \Big[\int \chi[(v - u_0) \cdot w]^2 \int \chi - \Big[\int \chi(v - u_0) \cdot w \Big]^2 \Big] \Big/ \Big(T^{1/2} \int \chi \Big)^2 \end{aligned}$$

which is positive (even for $r = 0$) by the Cauchy–Schwarz inequality. Finally, the invertibility of V just requires that

$$R(r) \to +\infty, \qquad r \to +\infty$$

which is readily proved by a simple analysis of the integrals which define R.

At this level, we have solved the first equation in (35) choosing u as

$$u = W(T, \beta)$$

and it remains to solve (with uniqueness) the second equation in (35), which can be written

$$\varphi(T) = |\beta|^2 + N\gamma \tag{36}$$

$$\varphi(T) := \int |v|^2 \chi \Big/ \int \chi \tag{37}$$

φ is C^1 and, using the chain rule,

$$\frac{d\varphi}{dT}=\frac{1}{2T^2Q\int\chi}\det\left[\int\chi(v)\,a_i(v)\,a_j(v)\,dv\right]_{1\leqslant i,j\leqslant 3}$$

where $\beta=u_0+Rw$, $|w|=1$, $R\geqslant 0$, $a_1=1$, $a_2=(v-u_0)\cdot w$, $a_3=|v-u_0|^2$, and

$$Q=\int[(v-u_0)\cdot w]^2\chi\int\chi-\left(\int\chi(v-u_0)\cdot w\right)^2>0$$

so that $d\varphi/dT>0$. Now, Eq. (36) is uniquely solvable if we can prove

$$\varphi(T)\to|\beta|^2,\qquad T\to 0 \tag{38}$$

$$\varphi(T)\to+\infty,\qquad T\to+\infty \tag{39}$$

But $\varphi(T)\geqslant(C_1/C_2)(|W(T,\beta)|^2+NT)$, which proves (39), and

$$\begin{aligned}|W(T,\beta)-\beta|&=\left|\int\chi(W-v)\right|\Big/\int\chi\\&\leqslant\int\chi\,|W-v|\Big/\int\chi\\&\leqslant(C_2/C_1)\sqrt{T}\int|v|\,e^{-|v|^2/2}\,dv/(2\pi)^{N/2}\end{aligned}$$

This implies that $W(T,\beta)\to\beta$ as $T\to 0$, and finally

$$\begin{aligned}0\leqslant\varphi(T)-|\beta|^2&=\int\chi\,|v-\beta|^2\Big/\int\chi\\&\leqslant 2\int\chi(|v-W|^2+|W-\beta|^2)\Big/\int\chi\\&\leqslant 2\frac{C_2}{C_1}NT+2\,|W-\beta|^2\end{aligned}$$

so that we get (38) and the proof of Proposition 2 is complete.

For further applications, let us notice that (by uniqueness mainly) the parameters (ρ, u, T) of the Maxwellian in (7) or (33) depend in a continuous fashion on (ρ_f, u_f, T_f) and α, β, γ as long as α remains positive.

We can now show that the Maxwellian in (33) can also be considered as a Gibbs equilibrium.

Theorem 3. The solution M of (33) is the unique minimum point of

$$\operatorname{Min}\left\{\int_{\mathbb{R}^N} \lambda(v)\, g(v) \ln g(v)\, dv;\ g(v) \geqslant 0,\right.$$

$$\left.\int \lambda g(1, v, |v|^2)\, dv = (\alpha, \alpha\beta, \alpha\,(|\beta|^2 + N\gamma))\right\}$$

We do not prove this theorem, which is a consequence of the convexity inequality

$$g \ln g - g - (M \ln M - M) \geqslant (g - M) \ln M$$

Let us also recall that, if $g \geqslant 0$ satisfies

$$\int (1 + |v|^2)\, g(v)\, dv < +\infty$$

then

$$\int g \ln^- g\, dv < +\infty$$

Finally, let us notice that the BGK model (3)–(7) satisfies a, formal for now, H-theorem

$$\partial_t \int f \ln f\, dv + \operatorname{div} \int v f \ln f\, dv \leqslant 0$$

because, as usual,

$$\int \lambda_f (f - M) \ln M\, dv = 0$$

4. GLOBAL EXISTENCE IN THE BGK MODEL

In the quadratic case $\nu = \rho_f$, the existence of solutions to the BGK model is still an open problem. For $\nu = 1$, we show how to extend the existence proof proposed in Perthame.[11] For bounded domains, the proof proposed by Ringeisen[15] or Perthame and Pham[12] could also be extended to our case.

Before stating our existence result, let us make precise the meaning of the set of equations (3)–(7). When $\rho_f(t, x) = 0$, the macroscopic quantities u, T, u_f, T_f are not defined and we just set $\lambda_f = C$, any constant, and $M = 0$, whatever are u and T. With this convention we have the following result.

Theorem 4. Under assumptions (9), (10), if the initial datum f_0 satisfies

$$f_0(x, v) \geqslant 0, \quad \int_{\mathbb{R}^{2N}} f_0(x, v)(1 + |x|^2 + |v|^2 + \ln^+ f_0)\, dx\, dv < \infty \tag{40}$$

there exists a nonnegative solution f of the BGK equations (3)–(7) with $\nu = 1$ which satisfies for some constant $C(T)$

$$\int_{\mathbb{R}^{2N}} f(t, x, v)(1 + |x|^2 + |v|^2 + \ln^+ f)\, dx\, dv \leqslant C(T), \quad \forall t \leqslant T \tag{41}$$

Remark. As usual, f, $|v|^2 f \in C([0, \infty[; L^1(\mathbb{R}^{2N}))$ and $\int |v|^3 f\, dv \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^N)$ (by the moments lemma). Also, we could obtain in the same way a stability result for weakly convergent initial data.

Proof of Theorem 4. We just indicate the modifications of the proof in ref. 11. The main new difficulty is to treat the vacuum in an approximate equation. The main steps are the following.

First, it is possible to build for any $\varepsilon > 0$ a solution $f_\varepsilon(t, x, v)$ of

$$\begin{gathered} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \lambda^\varepsilon f_\varepsilon = \lambda^\varepsilon M[f_\varepsilon] \\ f_\varepsilon(0, x, v) = f_0(x, v) \end{gathered} \tag{42}$$

where

$$\lambda^\varepsilon = \rho_{f_\varepsilon} \lambda_{f_\varepsilon} / (\varepsilon + \rho_{f_\varepsilon}) \tag{43}$$

and which satisfies the estimate (41).

We postpone the existence proof of f_ε and we indicate now how to pass to the limit as $\varepsilon \to 0$. As usual, extracting subsequences if necessary, f_ε converges to some $f(t, x, v)$ weakly in $L^1((0, T) \times \mathbb{R}^{2N})$. Moreover, $|v|^2 \lambda^\varepsilon M[f_\varepsilon]$ is bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{2N}))$, thus $\int |v|^3 f_\varepsilon\, dv$ is bounded in $L^1_{\text{loc}}((0, T) \times \mathbb{R}^N)$. Finally, using the variational principle of Section 3, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \lambda_{f_\varepsilon} M[f_\varepsilon] \ln M[f_\varepsilon]\, dx\, dv \\ &\qquad \leqslant \int_{\mathbb{R}^{2N}} \lambda_{f_\varepsilon} f_\varepsilon \ln f_\varepsilon\, dx\, dv \leqslant C(T), \qquad \forall t \leqslant T \end{aligned}$$

and with the assumption (9), we deduce that $M[f_\varepsilon]$ is also weakly compact in L^1. Hence, we can use the averaging lemma,[7] and thus $M[f_\varepsilon]$ converge a.e. and in $L^1((0, T) \times \mathbb{R}^{2N})$ to $M[f]$.

Also,

$$\lambda^\varepsilon \to \lambda_f \qquad \text{in} \quad L^1_{\text{loc}}(E) \text{ and a.e. in } E$$

where

$$E = \{(t, x, v) \in [0, T] \times \mathbb{R}^{2N};\, \rho_f(t, x) \neq 0\}$$

and this is enough to pass to the limit as $\varepsilon \to 0$ and to obtain a solution of (3)–(7). Indeed, on the set E^C, f_ε and $M[f_\varepsilon]$ converge in L^1 to 0; therefore λ^ε just needs to remain bounded, which is the case.

Let us now come back to the problem (42)–(43). A solution can be built using Schauder's theorem. Consider the set D of functions $g \in C([0, T]; L^1(\mathbb{R}^{2N}))$, $g \geqslant 0$, which satisfy for all $0 \leqslant t \leqslant T$

$$\int_{\mathbb{R}^{2N}} (1, |v|^2)\, g\, dx\, dv \leqslant e^{C_2 t} \int_{\mathbb{R}^{2N}} (1, |v|^2)\, f_0\, dx\, dv$$

$$\int_{\mathbb{R}^{2N}} |x|^2\, g\, dx\, dv \leqslant 2e^{C_2 t} \left(\int_{\mathbb{R}^{2N}} |x|^2 f_0 + T^2 e^{C_2 T} \int_{\mathbb{R}^{2N}} |v|^2 f_0 \right)$$

$$\int_{\mathbb{R}^{2N}} g \ln^+ g\, dx\, dv \leqslant C(t)$$

and

$$\partial_t g + v \cdot \nabla_x g = h, \qquad g(0, x, v) = f_0(x, v) \tag{44}$$

for some function $h(t, x, v)$ such that

$$\int_{\mathbb{R}^{2N}} (1 + |v|^2 + |x|^2 + \ln^+ |h|)\, |h|\, dx\, dv \leqslant C_0$$

It is easy to see that the constants $C(t)$, C_0 can be chosen such that $T_\varepsilon(D) \subset D$, where T_ε is the operator which associates to $g \in D$ the function $f = T_\varepsilon g \in C([0, T]; L^1(\mathbb{R}^{2N}))$, solution of

$$\begin{gathered} \partial_t f + v \cdot \nabla_x f + \lambda_g^\varepsilon f = \lambda_g^\varepsilon M[g] \quad \text{in} \quad [0, T] \times \mathbb{R}^{2N}, \\ f(0, x, v) = f_0(x, v) \end{gathered} \tag{45}$$

λ_g^ε is now defined [even on the set where $\rho_g(t, x) = 0$] by

$$\lambda_g^\varepsilon = \frac{\rho_g}{\varepsilon + \rho_g} \lambda_g \tag{46}$$

Notice that the solution f is explicitly given by the method of characteristics. It is now clear that $T_\varepsilon(D)$ is compact (for $\varepsilon > 0$ fixed) because the condition (44) and the moments lemma[11] applies, showing that the averages $(\rho_g, \rho_g u_g, \rho_g |u_g|^2 + N\rho_g T_g)_{g \in D}$ are compact in $L^1([0, T] \times B_R)$ for any $R > 0$.[7] As in ref. 11, this is enough for the compactness of T_ε, so that T_ε admits at least one fixed point. This completes the proof of Theorem 4.

5. EXISTENCE OF λ_0 FOR A GIVEN PRANDTL NUMBER IN]0, 1]

Proposition 5. For every $\Pr \in]0, 1]$, there is a function $\lambda_0(v)$ satisfying (9), (10) for some constants C_1, C_2 and the relations (11), (12) with the Prandtl number Pr.

Proof. We write $1/\lambda_0(v) = 1 + \chi(v)$, and we look for $\chi(|v|) \in L^\infty(\mathbb{R}^N)$, essinf $\chi > -1$,

$$\int \chi M_0(1, |v|^2, |v|^4, |v|^6)\, dv = \left(0, 0, 0, 2N(N+2)\left(\frac{1}{\Pr} - 1\right)\right)$$

Since $\chi = 0$ gives $\Pr = 1$, we only have to solve the problem for small values of Pr. Of course, the idea is to concentrate χ on the large values of $|v|$.

Let

$$V = \{\varphi(v) \in L^\infty(\mathbb{R}^N) \text{ s.t. } \varphi \text{ only depends on } |v|\}$$

V is an infinite-dimensional vector space, on which the linear forms

$$\varphi \to \int \varphi M_0\, |v|^{2m}\, dv \qquad (m = 0, 1, ...)$$

are independent.

First Step: $\forall \varepsilon > 0$, $\exists \varphi \in V$ s.t.

$$\varphi \geqslant 0, \qquad \int \varphi M_0(1, |v|^2, |v|^4)\, dv \leqslant \varepsilon, \qquad \int \varphi M_0\, |v|^6\, dv \geqslant \frac{1}{\varepsilon}$$

Proof. Choose $\varphi_0 \in V$, $\varphi_0 \geqslant 0$, $\varphi_0 \not\equiv 0$, supp $\varphi_0 \subset [1 < |v| < 2]$, and define $\varphi_\sigma = C_\sigma \varphi_0(v/\sigma)$, where C_σ is chosen so that

$$\int \varphi_\sigma M_0\, |v|^6 = \sigma$$

We have

$$\int \varphi_\sigma M_0 \, |v|^{2m} = C_\sigma \int \varphi_0 \left(\frac{v}{\sigma}\right) M_0 \, |v|^{2m} \sim C_\sigma \sigma^{2m} \int \varphi_0 \left(\frac{v}{\sigma}\right) M_0$$

$$\int \varphi_\sigma M_0 \, |v|^{2m} \sim \sigma^{2m-6} \int \varphi_\sigma M_0 \, |v|^6 \sim \sigma^{2m-5}$$

If we choose a large σ, we get a solution for the above problem.

Second Step: $\forall \varepsilon > 0, \exists \chi \in V$ s.t.

$$\chi \geqslant -\varepsilon, \quad \int \chi M_0(1, |v|^2, |v|^4) \, dv = 0, \quad \int \chi M_0 \, |v|^6 \, dv \geqslant \frac{1}{\varepsilon}$$

Proof. Let W be a four-dimensional subspace of V on which the four linear forms are independent (it is equivalent to asserting that W is a supplementary of the intersection of the kernels). For any I_0, I_2, I_4, I_6 there exists a corresponding $w \in W$ having the moments I_0, I_2, I_4, I_6, and if we choose $|I_0| \leqslant \eta, \ldots, |I_6| \leqslant \eta$, we get $\|w\|_{L^\infty} \leqslant \varepsilon$. By the first step there exists $\varphi \geqslant 0$ such that

$$\int \varphi M_0(1, |v|^2, |v|^4) \leqslant \eta, \qquad \int \varphi M_0 \, |v|^6 \geqslant 1/\varepsilon$$

Choose $w \in W$ such that

$$\int w M_0(1, |v|^2, |v|^4) \, dv = \int \varphi M_0(1, |v|^2, |v|^4) \, dv$$

$$\int w M_0 \, |v|^6 = 0$$

We have $\|w\|_{L^\infty} \leqslant \varepsilon$, and we may take $\chi = \varphi - w$. ∎

NOTE ADDED IN PROOF

We complete our references with the book by R. Brun, *Transport et Relaxation dans les Ecoulements Gazeux* (Masson, Paris, 1986), in which can be found the S-model of Shakov (p. 172), giving the right Prandtl number, but which does not satisfy the H-Theorem.

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